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THE THEORY OF FORCED CONVECTIVE HEAT TRANSFER IN BEDS OF FINE FIBRES—II

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INTRODUCTION

In Paper I of this series it has been shown that the conventional treatment of heat transfer in granular beds is not adequate for beds of textile fibres [1]. An alternative treatment is now examined which takes account of irregularities in the structure of the bed, and which in some respects gives results in closer agreement with experimental observations.

The situation considered is that of heat transfer between a semi-infinite slab of stationary fibres, and a stream of air which is forced through the assembly by an applied pressure.

SIMPLEST CASE

The most simplified model which is useful for later reference is as follows:

(1) The inter-fibre distance is small enough compared with the bed dimensions to allow the use of differential calculus.
(2) Heat capacities and densities of the air and the fibres are independent of temperature over the range considered.
(3) Heat transferred by conduction in the air in the direction of flow is negligible compared with that transferred by convection.
(4) The bed density and mean air velocity are uniform throughout the bed on any scale larger than a few inter-fibre distances.
(5) Heat transfer between air and fibres is so fast that the air and fibre temperatures are almost equal in any vicinity.

Consider a small volume element; then heat transferred to volume in any small time interval = heat gained by (textile + air) in that time.

Hence:

$$Sv\frac{\partial T}{\partial x} = (C + S)\frac{\partial T}{\partial t}.$$

In most cases of interest, $S \ll C$, and the equation becomes

$$Sv\frac{\partial T}{\partial x} + C\frac{\partial T}{\partial t} = 0. \tag{1}$$

The solution of this differential equation, with boundary conditions given by $T(o, t) = T_o(t)$ is

$$T(x, t) = T_o\left(t - \frac{x}{u}\right), \quad \text{where} \quad u = \frac{vS}{C}; \tag{2}$$

i.e. any temperature pattern (signal) imposed on the boundary is propagated unchanged down the bed with velocity u.

The two kinds of input which we wish to analyse are (a) step-function (b) sinusoidal.

In the former case let

$$T_o(t) = AU(t) = \begin{matrix} 0; t < 0 \\ A; t > 0 \end{matrix}$$

then

$$T(x, t) = AU\left(t - \frac{x}{u}\right).$$

In the latter case $T_o(t) = \sin \omega t$.

then $$T(x, t) = \sin \omega \left(t - \frac{x}{u}\right)$$

which is a progressive wave of wavelength

$$\lambda_o = \frac{2\pi u}{\omega}.$$

LOCAL HEAT TRANSFER

Previous treatments of granular beds (for example, that of Ledoux [2]) differ from the above simple case in that assumption (5) is replaced by a more realistic assumption. An air temperature T_A is introduced, in general different from the fibre temperature T_S, but related to it at any point by the heat-transfer equation

$$C \frac{\partial T_S}{\partial t} = h(T_A - T_S) \quad (3)$$

where h is the heat-transfer coefficient per unit volume of bed.

The resulting differential equation in T_S is

$$\frac{vSC}{h} \frac{\partial^2 T_S}{\partial x \partial t} + Sv \frac{\partial T_S}{\partial x} + C \frac{\partial T_S}{\partial t} = 0 \quad (4)$$

and this equation becomes identical with equation (1) if h is infinitely large.

As shown previously [1] the incorporation of a finite value of h leads to the attenuation of a sinusoidal signal, and also to an increase in the wavelength from the value λ_0 of the idealized model. However, these effects should be small for beds of textile fibres. This is demonstrated by Table 1 which shows the theoretical results for a range of air velocities and periods, the heat-transfer coefficient being assumed to be that applicable to cylinders in an airstream [3].

Table 1. Calculated effect of finite heat-transfer coefficient on sinusoidal temperature input

Period (s)	Air velocity (cm/s)	Fractional increase in wavelength	Attenuation after 15 cm
6	75	0·00009	0·81
20·3	77	0·000008	0·98
20·3	49	0·000011	0·96
20·3	31	0·000014	0·93
31	72	0·000004	0·99
31	52	0·000005	0·99
31	23	0·000007	0·96
61	58	0·0000011	0·997
61	38	0·0000014	0·99
61	21	0·000002	0·99

Bed of 18·5 μ terylene fibres.

However, experiments [1] show that a step-function input signal is broadened and a sine wave is attenuated by passage through the bed to a much greater extent than the calculations in Table 1 would suggest.

A new model is now examined in which account is taken of inhomogeneities in the bed by discarding assumption (4), of the ideal case. However, in the interests of simplicity, assumption (5) is restored, since Table 1 shows that for a bed of fine textile fibres h is effectively infinite for signals of period greater than a few seconds.

INHOMOGENEOUS BED

C and v are now assumed to vary from point to point, though not independently, since the air velocity tends to be smaller in dense regions. The air flow will have some of the features of turbulence because of the random variation of air velocity with the space co-ordinates. However, the flow is assumed to be laminar, and steady in time. The flow pattern will not depend on the velocity of the incoming air.

It is further assumed that the bed is homogeneous on a macroscopic scale, in the sense that it may be subdivided into smaller volumes, which will be identical in statistical properties. This means that if the results obtained from the model are to be applied to a finite bed, then the typical dimensions of clumps or other irregularities must be considerably smaller than the bed dimensions.

At the other end of the scale, the present model does not attempt to describe the effects of different local packing of the fibres, so that the irregularities considered are much larger than inter-fibre distances (about 5×10^{-3} cm). This means that v and C may be regarded as continuous functions of position. The justification is that the irregularities of smallest length scale are expected to have least effect on the behaviour of the bed.

BEHAVIOUR ON A STREAMLINE

Let P be a point somewhere in the bed, and let s be the distance from the boundary to P, measured along the streamline passing through P.

Then we assume that the idealized equation (1) holds along the streamline, i.e.

$$Sv\frac{\partial T}{\partial s} + C\frac{\partial T}{\partial t} = 0,$$

where the local values of v and C are used.

At this stage we assume that lateral heat conduction is negligible. The effect of this assumption will be examined later.

Then any disturbance is propagated along a streamline with a changing velocity u given by the value of vS/C at each point. In the succeeding analysis we consider the effect of a variation throughout the bed of the signal velocity vS/C. The distribution of this quantity cannot be obtained from the distribution of packing density without considering the negative correlation between v and C, which is not attempted here.

The time τ taken for a signal to travel from the boundary to a point P is given by the integral along the streamline

$$\tau = \int_0^P \frac{ds}{u} = \int_0^P \frac{C}{Sv}\, ds.$$

If the flow is laminar, the velocity at any point is proportional to V, the velocity of the air entering the bed, i.e. $v = kV$, and k is constant at each point.

Then

$$\tau = \frac{1}{V}\int_0^P \frac{C}{Sk}\, ds.$$

Denote the line integral in the expression by l, then $\tau = l/V$. In the air upstream from the bed, signals move with the air speed V. Hence l may be regarded as the equivalent path length (in air) of the path considered in the bed.

The temperature T at any point in the bed will now be determined by its equivalent path length from the boundary, and will be equal to the temperature at the boundary at a time l/V seconds earlier.

i.e. $$T = T_0\left(t - \frac{l}{V}\right) \tag{5}$$

where $T_0(t)$ is the input temperature.

DISTRIBUTION OF PATH LENGTHS

If a streamline is divided into consecutive segments, then the equivalent path lengths of adjacent segments will be independent of each other, provided that each segment is longer than the typical dimensions of irregularities in the bed.

Let the bed be divided into section by planes parallel to the boundary and separated by a distance a, large enough for the above condition to hold (see Fig. 1). Let λ_a be the equivalent length of a typical segment of a streamline, lying between consecutive planes.

Then λ_a will be randomly distributed with a frequency function $f(a, l)$. That is, the probability that λ_a lies between l and $l + dl$ is $f(a, l)\, dl$. The

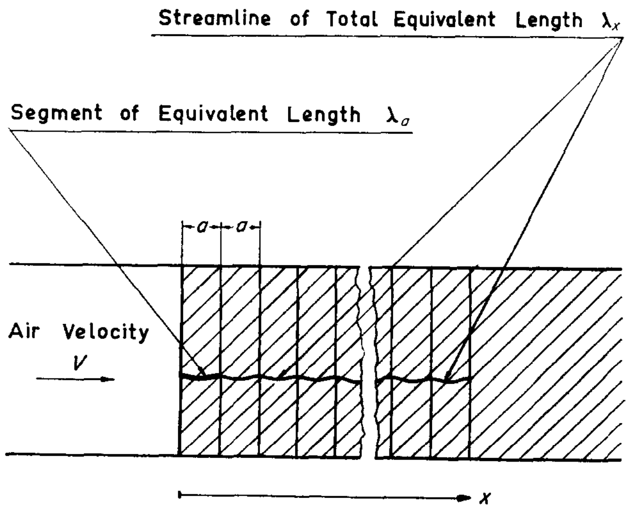

FIG. 1. Longitudinal section through bed.

variability in λ_a will be due partly to the variation in the geometrical lengths of paths, and partly to the variation in speed with which a signal travels along a streamline.

The behaviour of the bed will now be determined by the function $f(a, l)$. Denote the mean or first moment of $f(a, l)$ by $m_1(a)$. Then $\rho = [m_1(a)/a]$ is the mean equivalent path length per unit thickness of bed. The second moment is denoted by $m_2(a) = \sigma_a^2$ and the third moment by $m_3(a)$.

If we now consider a plane in the bed, distant x from the boundary, we can determine the total equivalent path length λ_x along a streamline between the boundary and any point on the plane. If the determinations are made at each point on the plane on a grid of mesh equal to at least a, then these values of λ_x will be independent and distributed with a frequency function which we denote by $f(x, l)$. Since the quantity λ_x is made up of n independent quantities λ_a where $n = x/a$, and λ_a is randomly distributed according to the function $f(a, l)$ then the distribution (x, l) can be derived from the distribution of $f(a, l)$. In particular, the first three moments of $f(x, l)$ are given by n times the corresponding moment of $f(a, l)$ [4]. Thus for $x > a$, these moments should increase linearly with x. In an easily understood notation we can write

$$m_1(x) = \frac{x}{a} m_1(a) = \rho x$$

$$m_2(x) = \frac{x}{a} \sigma_a^2 \quad = \beta x.$$

The mean or expected value of the temperature at the plane x, and at time t, $\bar{T}(x, t)$, can now be found, using equation (5). Since the temperature at a point is determined by λ and t, and since a path length between l and $l + \mathrm{d}l$ occurs with probability $f(x, l)\,\mathrm{d}l$ then the weighted mean or expected value of temperature is given by

$$\bar{T}(x, t) = \int_{-\infty}^{\infty} f(x, l)\, T(l, t)\, \mathrm{d}l$$

$$= \int_{-\infty}^{\infty} f(x, l)\, T_o\left(t - \frac{l}{V}\right) \mathrm{d}l.$$

In practice, $f(x, l)$ is zero for negative values of l, and hence

$$\bar{T}(x, t) = \int_{0}^{\infty} f(x, l)\, T_o\left(t - \frac{l}{V}\right) \mathrm{d}l. \tag{6}$$

$\bar{T}$ can be derived not only in terms of the input temperature conditions, as above, but also in terms of the mean temperature behaviour at any plane upstream from x.

Consider two adjacent slabs in the bed, of thickness d_1 and d_2, the combined slab having a thickness $d = d_1 + d_2$.

Let a typical streamline have equivalent path lengths l_1 and l_2 in the two slabs, the combined path length being l. Then by the law of addition of two independent variables [4]

$$f(d, l) = \int_{-\infty}^{\infty} f(d_1, l_1) f(d_2, l - l_1)\, \mathrm{d}l_1 \tag{7}$$

since l_1 and l_2 are independent.

From equation (6) above

$$\bar{T}(d, t) = \int_{-\infty}^{\infty} f(d, l)\, T_o\left(t - \frac{l}{V}\right) \mathrm{d}l$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(d_1, l_1) f(d_2, l - l_1)\, T_o\left(t - \frac{l}{V}\right) \mathrm{d}l_1\, \mathrm{d}l$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(d_1, l_1) f(d_2, l_2)\, T_o\left(t - \frac{l_1 + l_2}{V}\right) \mathrm{d}l_1\, \mathrm{d}l_2$$

since we may integrate over all values of l_1 and l_2, instead of using l_1 and l. By equation (6)

$$\int_{-\infty}^{\infty} f(d_1, l_1)\, T_o\left(t - \frac{l_1}{V}\right) \mathrm{d}l_1 = \bar{T}(d_1, t).$$

Hence the above expression becomes

$$\bar{T}(d, t) = \int_{-\infty}^{\infty} f(d_2, l_2)\, \bar{T}\left(d_1, t - \frac{l_2}{V}\right) \mathrm{d}l_2.$$

If complete mixing were to take place at the plane separating the two slabs ($x = d_1$), so that all air entering the second slab would be at the mean temperature $\bar{T}(d_1, t)$, then the resulting value of $\bar{T}(d_1 + d_2, t)$ would be exactly the same as $\bar{T}(d, t)$ above. Thus complete lateral mixing at any plane in the bed has no effect on the mean temperature downstream. This justifies our neglect of lateral conduction or mixing.

The reason why lateral conduction plays no part in this model is that the speed of a signal varies randomly along the length of a single streamline. Hence a signal transferred rapidly, say, from the boundary to a particular cross section along a certain streamline, may be transferred either rapidly or slowly to further cross sections, and the probabilities will not be changed by conduction to adjacent streamlines.

The situation differs in this respect from that of convective dispersion in a cylindrical pipe [5], where the velocity along a single streamline will be consistent along its whole length, resulting in a dispersive mechanism which is retarded by lateral transfer.

STEP-FUNCTION INPUT

If

$$T_0(t) = AU(t) = \begin{matrix} 0; t < 0 \\ A; t > 0 \end{matrix}$$

then

$$\bar{T}(x, t) = \int_0^\infty f(x, l)\left(t - \frac{l}{V}\right) \mathrm{d}l$$

$$= A \int_0^{Vt} f(x, l)\, \mathrm{d}l. \quad (8)$$

Now the probability that a particular path terminating at x has an equivalent length less than l is given by the cumulative distribution function, which we denote by $F(x, l)$, where $F(x, l) = \int_0^l f(x, l')\, \mathrm{d}l'$. Hence the measurement of $\bar{T}(x, t)$ gives an estimate of $A\, F(x, Vt)$. If $\bar{T}$ at any plane is plotted against time, and the resultant curves differentiated, a direct measure of $f(x, l)$ is obtained. Fig. 2 shows $f(x, l)$ as estimated in this way at three different values of x, using the equipment described in Part I, and using an air speed through the bed of 10 cm/s.

SINUSOIDAL TEMPERATURE INPUT

If $T_0(t)$ is the real part of $A\, e^{i\omega t}$, then

$$\bar{T}(x, t) = A \int_{-\infty}^{\infty} f(x, l)\, e^{i\omega[t-(l/V)]}\, \mathrm{d}l$$

$$= A\, e^{i\omega t} \int_{-\infty}^{\infty} f(x, l)\, e^{-i(\omega l/V)}\, \mathrm{d}l. \quad (9)$$

Hence $\bar{T}$ at any plane will be a sinusoidal function of time with the frequency of the input signal, and with phase and amplitude given by the integral in the above expression, which we will denote by $B(x)\, e^{-i\theta(x)}$.

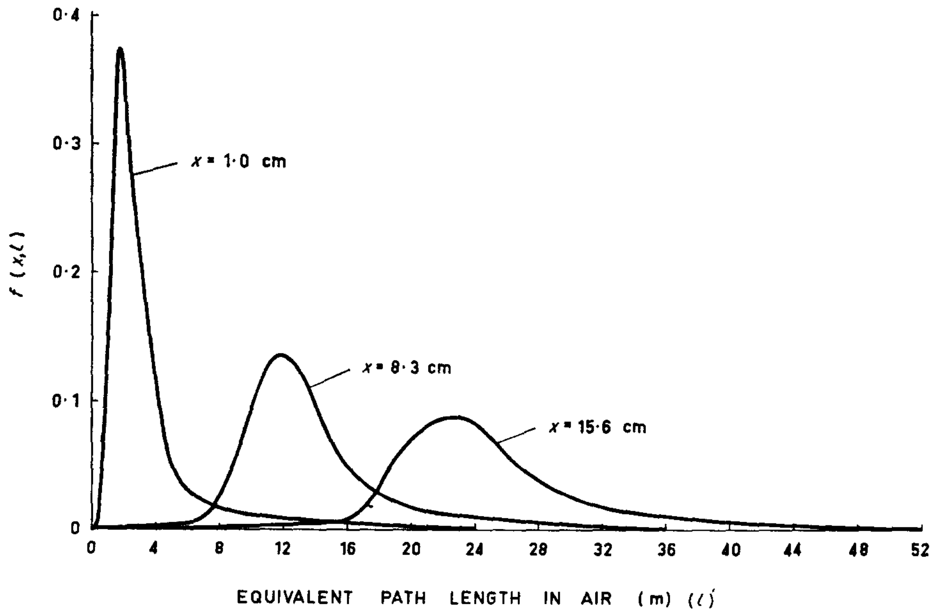

FIG. 2. Estimates of path-length distributions in bed of terylene fibres, obtained from behaviour with step-function temperature input.

The amplitude $B(x)$ is given by

$$B^2(x) = \left[\int_{-\infty}^{\infty} f(x, l) \cos \frac{\omega l}{V} \, dl\right]^2 + \left[\int_{-\infty}^{\infty} f(x, l) \sin \frac{\omega l}{V} \, dl\right]^2$$

and $\theta(x)$ is determined by the ratio of the imaginary and real parts

$$\tan \theta_x = \frac{\int_{-\infty}^{\infty} f(x, l) \sin \frac{\omega l}{V} \, dl}{\int_{-\infty}^{\infty} f(x, l) \cos \frac{\omega l}{V} \, dl}.$$

Because of the particular way in which $f(x, l)$ depends on x, it can be shown that $\theta(x)$ increases linearly with x. Let a typical streamline have an equivalent length l_1 between the planes $x = 0$ and $x = d_1$, and an equivalent length l_2 between the planes $x = d_1$ and $x = d_1 + d_2$. Let $l_1 + l_2 = l$. Then

$$B(d_1 + d_2) e^{-i\theta(d_1+d_2)} = \int_{-\infty}^{\infty} f(d_1 + d_2 l) e^{-i(\omega l/V)} \, dl.$$

Note that this expression is a Fourier transform of the function $f(d_1 + d_2, l)$ which is itself a faltung of the functions $f(d_1, l)$ and $f(d_2, l)$.

Hence by a well-known theorem [6],

$$B(d_1 + d_2) e^{-i\theta(d_1+d_2)} = B(d_1) e^{-i\theta(d_1)} . B(d_2) e^{-i\theta(d_2)}.$$

Thus the phase angle at any time changes linearly with x, and the fractional change in amplitude per unit change in x is constant down the bed. These results show that a sinusoidal input produces a sinusoidal progressive wave for $\bar{T}$ within the bed, whose amplitude decreases exponentially with x.

$$\bar{T}(x, t) = A e^{-ax} \sin\left(\omega t - \frac{2\pi x}{\lambda}\right).$$

Put $x = 1$; then

$$a = -\tfrac{1}{2} \log \left[\left(\int f(1, l) \cos \frac{\omega l}{V} \, dl\right)^2 + \left(\int f(1, l) \sin \frac{\omega l}{V} \, dl\right)^2\right]$$

and

$$\frac{2\pi}{\lambda} = \tan^{-1}\left[\frac{\int f(1, l) \sin (\omega l/V) \, dl}{\int f(1, l) \cos (\omega l/V) \, dl}\right].$$

In the special case of a uniform bed, in which a slab of unit thickness contains paths all of equal equivalent length C/S

$$f(1, l) = 0, \; l \neq \frac{C}{S}$$

and

$$\int_{-\infty}^{\infty} f(1, l) \, dl = 1.$$

In this case substitution above shows that $a = 0$ and $(2\pi/\lambda) = (\omega C/SV)$, and the results reduce to those of the idealized case.

It is worth noting that the above expressions for b and λ involve the characteristics of the bed and the parameter ω/V which depends on the input conditions.

Now

$$\frac{\omega}{V} = \frac{2\pi}{\text{wavelength in air}};$$

hence both the wavelength and attenuation in the bed depend only on the wavelength of the input signal in air, and it follows that the attenuation will be determined by the wavelength measured in the bed.

DISCUSSION

The model described here was developed in order to explain the results obtained in Paper I [1], and is believed to present a truer picture of the physical conditions governing heat transfer in textile beds. The predictions of the model may be tested against the data of Part I as follows:

(1) This model gives a plausible explanation of why heat transfer appears less efficient than expected from existing theory [1]. The previous treatment fails, not because the single fibre behaviour is different from that in an air stream, but because the bed as a whole does not conform to the established model.

(2) This model predicts that the attenuation with distance of a sinusoidal input signal in a

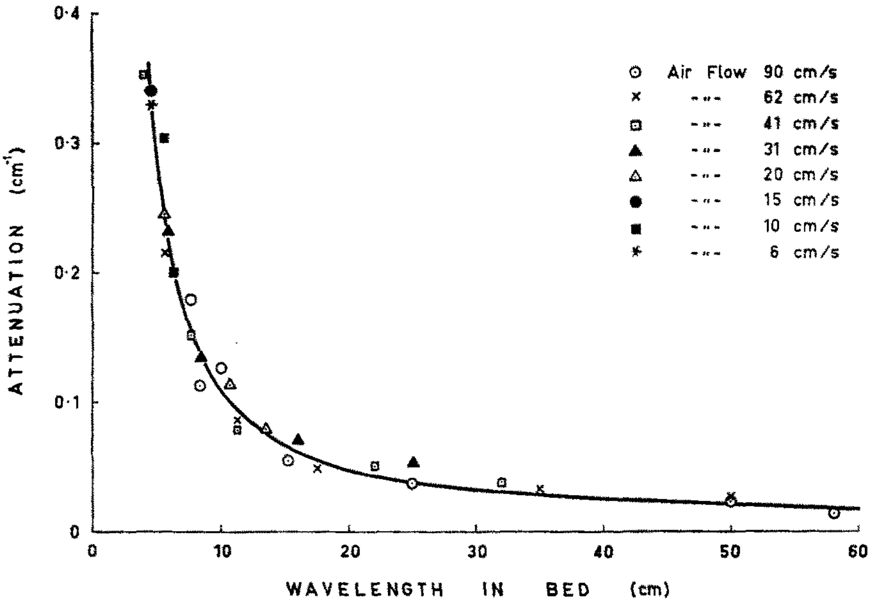

FIG. 3. Relation between attenuation and wavelength for sinusoidal input.

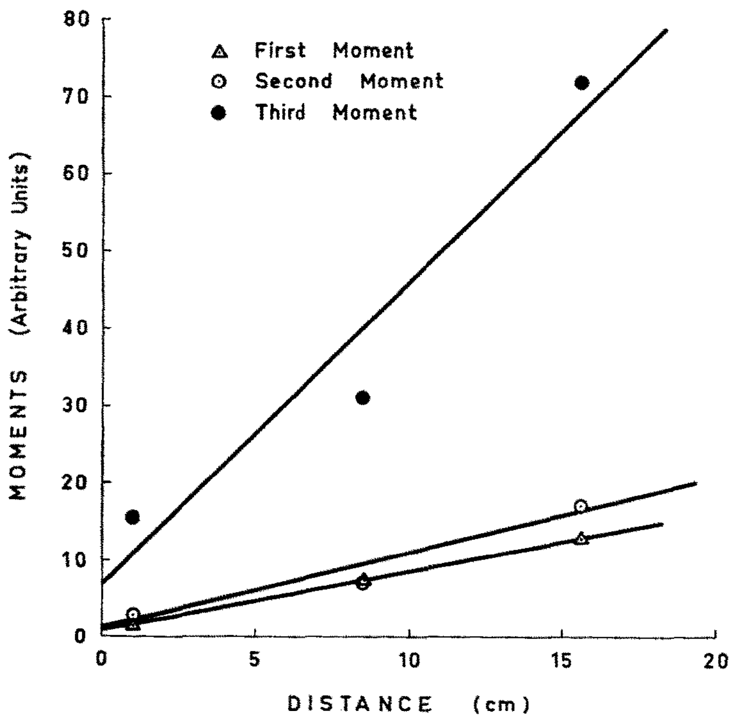

FIG. 4. Apparent behaviour of moments.

2H

particular bed depends only on the wavelength in the bed. The attenuation for each experiment is plotted against wavelength in Fig. 3, and it can be seen that the points lie close to an single curve, as predicted. The existing theory predicts a dependence on h, Sv and ω as in equation (12) of Part 1, which was not borne out by experiment. It was found necessary to make h approximately proportional to $1/\omega$ for a given air flow, in order to fit the data.

(3) If a step function temperature input is applied, the measurement of $\bar{T}$ at a plane allows us to calculate the distribution function as in Fig. 2, and its moments. Hence we can test the prediction that the first three moments increase linearly with x. Fig. 4 shows the moments plotted against x, and it can be seen that the prediction is not borne out too well for the second and third moment. There are several possible reasons for this.

(a) The input was a poor step-function, and had a long tail which made estimation of the higher moments inaccurate.

(b) The test was made at low air flow, so that the original assumption (3) may not hold, and longitudinal conductivity in the air could be significantly large compared with heat transferred by convection.

SUMMARY

The view presented here is that the overall heat transfer in beds of textile fibres is governed by irregularities in the flow. According to this view, heat transfer to individual fibres is so rapid that the heat-transfer coefficient has only a secondary influence on the large scale behaviour of the bed. Instead, the picture is that of a signal entering the bed and losing coherence due to the inhomogeneity of the medium through which it passes. The precise manner of dependence on the bed structure has not yet been investigated. This mechanism may be of some importance in other situations where either local heat or mass transfer [2, 7], or longitudinal conductivity [8] has previously been assumed to be the sole important factor.

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